Pricing Fixed-Income Derivatives with the Forward-Risk Adjusted Measure

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1 The problem

Consider a fixed-income derivative with a single payoff at time T which depends on the term structure. In particular, we will look at options on zero-coupon bonds and interest-rate caps. For a call option on a zero-coupon bond maturing at time T_1, the time T payoff — and hence value of the derivative — is given by

$$V_T = \max\Big(P(T, T_1) - K,\, 0 \Big). \tag{1}$$

By the no-arbitrage theorem, the price today ($t = 0$) is

$$V_0 = E_0^Q \left[e^{-\int_0^T r_s ds} V_T \right], \tag{2}$$

where the expectation is taken under the risk-neutral distribution (also called the Q-measure). Thus, the price depends on the stochastic process for the short rate and the contractual specification of the security (i.e., how the payoff is linked to the term structure).

The price V_0 in equation (2) is given by the expectation of the product of two dependent random variables, and calculating this expectation is often quite difficult. The purpose of this note is presenting a change-of-measure technique which considerably simplifies the evaluation of V_0. Specifically, we are going to calculate V_0 as

$$V_0 = P(0, T) E_0^{Q^T}(V_T), \tag{3}$$

where Q^T is a new probability measure (distribution), the so-called *forward-risk* adjusted measure. This technique was introduced in the fixed-income literature by Jamshidian (1991).

2 Model setup and notation

Our term-structure model is a general one-factor HJM model, see Heath, Jarrow and Morton (1992) or Lund (1998) for an exposition. Under the Q-measure, forward rates are governed by

$$df(t, T) = -\sigma(t, T)\sigma_P(t, T)dt + \sigma(t, T)dW_t^Q, \tag{4}$$

where

$$\sigma_P(t, T) = -\int_t^T \sigma(t, u)du. \tag{5}$$

Bond prices evolve according to the SDE

$$dP(t, T) = r_t P(t, T)dt + \sigma_P(t, T)P(t, T)dW_t^Q, \tag{6}$$

so $\sigma_P(t, T)$ is the time t volatility of the zero maturing at time T.

3 The forward-risk adjusted measure

Under certain regularity conditions, the price of the derivative security follows the SDE

$$dV_t = r_t V_t dt + \sigma_V(t) V_t dW_t^Q. \tag{7}$$

This means that, under the risk-neutral distribution, the expected rate of return equals the short rate (just like any other security), and the return volatility is $\sigma_V(t)$. So far, neither V_t nor $\sigma_V(t)$ are known, but this is not essential for the following arguments. In fact, the only thing that matters is that the price process has the *form* (7) since this facilitates pricing by the forward-risk adjusted measure.

We begin by defining the deflated price process

$$F_t \equiv V_t \,/\, P(t,T) \tag{8}$$

for $t \in [0,T]$. We can interpret F_t as the price of V_t in units of the T-maturity bond price (i.e., as a relative price). Using Ito's lemma, it can be shown that

$$dF_t = \sigma_P(\sigma_P - \sigma_V)F_t dt + (\sigma_V - \sigma_P)F_t dW_t^Q \tag{9}$$

where σ_P and σ_V are shorthand notation for $\sigma_P(t,T)$ and $\sigma_V(t)$, respectively. The proof of (9) is given in appendix A.

Furthermore, we define a new probability measure, Q^T, such that

$$W_t^{Q^T} = W_t^Q - \int_0^t \sigma_P(u,T)du, \quad t \in [0,T], \tag{10}$$

is a Brownian motion under Q^T.[1] In differential form the relationship between $W_t^{Q^T}$ and W_t^Q is

$$dW_t^{Q^T} = dW_t^Q - \sigma_P(t,T)dt = dW_t^Q - \sigma_P dt. \tag{12}$$

The new probability measure is known as the forward-risk adjusted measure. It is very important to note that there is a different measure for each T (payoff date).

If we substitute (12) into (9), we obtain the dynamics of F_t under the new probability measure Q^T. Straightforward calculations give

$$\begin{aligned} dF_t &= -\sigma_P(\sigma_V - \sigma_P)F_t dt + (\sigma_V - \sigma_P)F_t \left(dW_t^{Q^T} + \sigma_P dt\right) \\ &= (\sigma_V - \sigma_P)F_t dW_t^{Q^T}, \end{aligned} \tag{13}$$

[1] We have, implicitly, used a similar technique when defining the risk-neutral measure earlier. Specifically, if W_t is a Brownian motion under the original (true) probability measure, we define Q such that

$$W_t^Q = W_t^Q + \int_0^t \lambda(u)du \tag{11}$$

is a Brownian motion under Q. Note that $\lambda(u)$ is the market price.

as the two terms with dt cancel out. Thus under Q^T, the drift is zero and F_t is a martingale. The new probability measure was defined in order to obtain this result since the martingale property implies that

$$F_t = E_t^{Q^T}(F_T). \tag{14}$$

Moreover, by definition $P(T,T) = 1$, so at maturity we have $F_T = V_T$, and using (14), the current ($t = 0$) price of the derivative security can now be calculated as

$$\begin{aligned} V_0 &= P(0,T)F_0 = P(0,T)E_0^{Q^T}(F_T) \\ &= P(0,T)E_0^{Q^T}(V_T), \end{aligned} \tag{15}$$

which is $P(0,T)$ times the expected payoff under Q^T. Generally, the latter calculation is a lot simpler than direct evaluation of the expectation under Q, as in equation (2) above. With (15) at hand, the only remaining task is determining the distribution of the payoff under the forward-risk adjusted measure.[2]

We conclude this section by noting that $f(t,T)$ is a martingale under Q^T. To see this, substitute (12) into the forward-rate SDE (4),

$$\begin{aligned} df(t,T) &= -\sigma(t,T)\sigma_P(t,T)dt + \sigma(t,T)\left(dW_t^{Q^T} + \sigma_P(t,T)dt\right) \\ &= \sigma(t,T)dW_t^{Q_T}. \end{aligned} \tag{16}$$

This property turns out be very useful when pricing at-the-money interest-rate caps, cf. the second example in the next section.

4 Two examples

For concreteness, we use the extended Vasicek model which is a special case of the one-factor HJM model with

$$\sigma(t,T) = \sigma e^{-\kappa(T-t)}, \tag{17}$$

and

$$\sigma_P(t,T) = -\sigma \int_t^T \sigma(t,u)du = \sigma \frac{e^{-\kappa(T-t)} - 1}{\kappa}. \tag{18}$$

The extended Vasicek model is a Markovian HJM model, cf. Lund (1998), but the following pricing formulas for bond options [equation (30)] and interest-rate caps [equation (40)] do not depend on the Markov property.

[2] Note that we are using the martingale property in the "opposite" direction (i.e., backwards) in equations (14) and (15). Normally, we know F_t and use the martingale property to compute the expected value at time T. This line of reasoning is implicit in the weak form of market efficiency [Fama (1970)] where we argue that the best forecast of the future stock price is the stock price today. In the context of pricing derivatives, we use our knowledge about the Q^T-distribution of the payoff V_T, combined with the martingale property of F_t (the relative price), to compute the current (relative) price of the derivative security.

4.1 Call option on a zero-coupon bond

In the first example, the fixed-income derivative is a call option on a zero-coupon bond maturity at time T_1. The option expires (matures) at time $T < T_1$ with the following payoff:

$$C_T = \max\Big(P(T, T_1) - K,\, 0 \Big), \tag{19}$$

where K is the strike (exercise) price of the option.

In order to price this security, we need the distribution of C_T under the forward-risk adjusted measure. Since C_T only depends $P(T, T_1)$ and since $P(T,T) = 1$, we can calculate the expectation of C_T from the distribution of the relative price,

$$F(t, T, T_1) = P(t, T_1)/P(t, T), \tag{20}$$

which is also the forward price of the T_1-maturity bond for delivery at time T. Using the results of section 3, the SDE for $F(t, T, T_1)$ under Q^T is given by

$$\begin{aligned} dF(t,T,T_1) &= \{\sigma_P(t,T_1) - \sigma_P(t,T)\}\, F(t,T,T_1) dW_t^{Q^T} \\ &\equiv \sigma_F(t,T,T_1) F(t,T,T_1) dW_t^{Q^T}. \end{aligned} \tag{21}$$

Since bond prices are always strictly positive, the logarithm of $F(t, T, T_1)$ is well-defined, and a simple application of Ito's lemma gives

$$d \log F(t,T,T_1) = -\frac{1}{2}\sigma_F^2(t,T,T_1)dt + \sigma_F(t,T,T_1)dW_t^{Q^T}. \tag{22}$$

After integrating from $t = 0$ to $t = T$ we have

$$\begin{aligned} \log F(T,T,T_1) &= \log P(T,T_1) = \\ \log F(0,T,T_1) &- \frac{1}{2}\int_0^T \sigma_F^2(t,T,T_1)dt + \int_0^T \sigma_F(t,T,T_1)dW_t^{Q^T}. \end{aligned} \tag{23}$$

The first equality in (23) follows because $P(T, T) = 1$. Moreover, if $\sigma_F(t, T, T_1)$ is deterministic, i.e. if the model is Gaussian, it follows from (23) that $\log P(T, T_1)$ is conditionally normally distributed with variance

$$\omega_F^2(T,T_1) = \int_0^T \sigma_F^2(t,T,T_1)dt, \tag{24}$$

and mean

$$\begin{aligned} \mu_F(T,T_1) &= \log F(0,T,T_1) - \frac{1}{2}\int_0^T \sigma_F^2(t,T,T_1)dt \\ &= \log F(0,T,T_1) - \frac{1}{2}\omega_F^2(T,T_1). \end{aligned} \tag{25}$$

For the extended Vasicek model, $\sigma_F(t, T, T_1)$ is given by

$$\begin{aligned} \sigma_F(t,T,T_1) &= \frac{\sigma}{\kappa}\left(e^{-\kappa(T_1 - t)} - e^{-\kappa(T-t)}\right) \\ &= \frac{\sigma}{\kappa}e^{-\kappa(T-t)}\left(e^{-\kappa(T_1 - T)} - 1\right), \end{aligned} \tag{26}$$

cf. equation (18), and the variance $\omega_F^2(T, T_1)$ can be calculated as

$$\begin{aligned}
\omega_F^2(T,T_1) &= \frac{\sigma^2}{\kappa^2}\left(e^{-\kappa(T_1 - T)} - 1\right)^2 \int_0^T e^{-2\kappa(T - t)} dt \\
&= \left(\frac{e^{-\kappa(T_1 - T)} - 1}{\kappa}\right)^2 \times \left(\sigma^2 \frac{1 - e^{-2\kappa T}}{2\kappa}\right) \\
&= B^2(T_1 - T) \cdot \mathrm{Var}_0^{Q^T}(r_T), \qquad (27)
\end{aligned}$$

since the last parenthesis in the second line can be recognized as the conditional variance of r_T, see equation (4) in Jamshidian (1989). The function $B(\tau)$ is the "factor loading" (stochastic duration) for the Vasicek model.[3] Finally, the price of the call option, denoted $C(T, K)$, is given by:

$$\begin{aligned}
C(T,K) &= P(0,T)E_0^T(C_T) \\
&= P(0,T)\int_{\log K}^{\infty} \left(e^x - K\right) \frac{1}{\sqrt{2\pi}\omega_F} e^{-(x - \mu_F)^2/2\omega_F^2}\, dx \\
&= P(0,T)\int_{\log K}^{\infty} e^x \frac{1}{\sqrt{2\pi}\omega_F} e^{-(x - \mu_F)^2/2\omega_F^2}\, dx \\
&\quad -P(0,T)K\int_{\log K}^{\infty} \frac{1}{\sqrt{2\pi}\omega_F} e^{-(x - \mu_F)^2/2\omega_F^2}\, dx \\
&= P(0,T_1)N(d_1) - P(0,T)KN(d_2), \qquad (30)
\end{aligned}$$

where $N(\cdot)$ is the cumulative normal distribution function, ω_F is shorthand notation for $\omega(T, T_1)$, and

$$d_1 = \left(\log\frac{P(0,T_1)}{P(0,T)} - \log K + \frac{1}{2}\omega_F^2\right) / \,\omega_F \qquad (31)$$

$$d_2 = d_1 - \omega_F. \qquad (32)$$

The calculation is completely analogous to the Black-Scholes model for call options on stock prices, so we skip the intermediate steps leading to the final expression for $C(T, K)$ in equation (30).[4]

[3] An alternative derivation for the extended Vasicek model can be based on the formula

$$\log P(T,T_1) = A(T,T_1) + B(T_1 - T)r_T, \qquad (28)$$

cf. Lund (1998), and since $A(T, T_1)$ is deterministic, the variance of $\log P(T, T_1)$ under Q^T is

$$\mathrm{Var}_0^{Q^T}(\log P(T,T_1)) = B^2(T_1 - T)\cdot \mathrm{Var}_0^{Q^T}(r_T). \qquad (29)$$

Of course, with this approach we would still need to determine the variance of r_T.

[4] Hint for your own derivation: if x is $\mathcal{N}(\mu, \sigma)$, the truncated mean of $\exp(x)$ is

$$\int_L^{\infty} e^x \frac{1}{\sqrt{2\pi}\sigma} e^{-(x - \mu)^2/2\sigma^2} dx$$

4.2 Interest-rate caps

Consider a derivative with the following payoff at time T:

$$C_T = \max(r_T - K, 0). \tag{34}$$

This corresponds to a simple interest-rate cap.[5] The price (today) of the cap is given by:

$$C(T,K) \;=\; P(0,T)E_0^{Q^T}(C_T) \;=\; P(0,T)E_0^{Q^T}\left[\max(r_T - K, 0)\right]. \tag{35}$$

In order to calculate (35) we must determine the distribution of r_T under Q^T (the forward-risk adjusted measure). First, note that

$$r_T = f(T,T), \tag{36}$$

so we can obtain the distribution of r_T from $f(T,T)$.

Second, under Q^T the T-maturity forward rate is a martingale, as shown in equation (16) in section 3. This means that

$$r_T \;=\; f(T,T) \;=\; f(0,T) + \int_0^T \sigma(t,T)dW_t^{Q^T}. \tag{37}$$

If $\sigma(t,T)$ is deterministic, r_T is conditionally normally distributed (at time $t = 0$) with mean $f(0,T)$ and variance

$$\mathrm{Var}_0^{Q^T}(r_T) = \int_0^T \sigma^2(t,T)dt \equiv v^2(0,T). \tag{38}$$

For the extended Vasicek model, this becomes

$$v^2(0,T) = \sigma^2 \int_0^T e^{-2\kappa(T-t)}dt = \sigma^2 \frac{1 - e^{-2\kappa T}}{2\kappa}. \tag{39}$$

$$= \int_L^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{\mu + \frac{1}{2}\sigma^2 - (x - \mu - \sigma^2)^2/2\sigma^2} dx$$

$$= \exp\left(\mu + \frac{1}{2}\sigma^2\right) N\left[\frac{-L + (\mu + \sigma^2)}{\sigma}\right], \tag{33}$$

since the integrand in the second line equals $\exp\left(\mu + \frac{1}{2}\sigma^2\right)$ times the density function of a normal distribution with mean $\mu + \sigma^2$ and variance σ^2.

[5] In the real world, caps are more complicated. The underlying interest rate is not the short rate, but (say) the three-month (LIBOR) interest rate. Moreover, a cap contract for T years on the three-month rate is a portfolio of $4T$ so-called caplets (single-payment caps), and the payment of the i'th caplet is $0.25\max[R_{3M}((i-1)/4) - K,\ 0)]$, where $R_{3M}(t)$ is the three-month interest rate at time t. The payments are made in arrear, which means that the i'th payment is made at time $t = i/4$ (three months after the fixing date). Of course, the "real world" cap can be priced by the same principles as the simple cap described in this section, that is by a suitable application of the the forward-risk adjusted measure for each caplet. Needless to say, the algebra become more involved, but that is the only real difference.

Finally, we compute the expected payoff under Q_T and hence the price of the cap. For reason of space, we concentrate on the at-the-money cap[6] where $K = f(0,T)$, and

$$\begin{aligned} C(T, f(0,T)) &= P(0,T) E_0^{Q^T}\left[\max(r_T - f(0,T), 0)\right] \\ &= P(0,T)\frac{v(0,T)}{\sqrt{2\pi}}, \end{aligned} \tag{40}$$

where $v(0,T)$ is defined in (38) for any Gaussian one-factor HJM model, and in (39) for the extended Vasicek model. The second line in (40) follows by noting that the payoff can be written as

$$\max(r_T - f(0,T), 0) = \max\left(\int_0^T \sigma(t,T) dW_t^{Q^T}, 0\right) \tag{41}$$

which has the same distribution as

$$\sqrt{\int_0^T \sigma^2(t,T)dt} \times \max(x,0) = v(0,T)\max(x,0), \tag{42}$$

where x is $\mathcal{N}(0,1)$, that is normally distributed with zero mean and unit variance. The expected value of $\max(x,0)$ is given by

$$\begin{aligned} E[\max(x,0)] &= \int_0^\infty \frac{x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-u} du \\ &= \frac{1}{\sqrt{2\pi}}. \end{aligned} \tag{43}$$

The second line follows by a change of variables from x to $u = \frac{1}{2}x^2$. This completes the proof of (40).

[6]The price formula for a cap with arbitrary exercise price, K, (in the extended Vasicek model) can be found in Longstaff (1995).

Appendix A: proof of equation (9)

To simplify the notation, we write the SDE for V_t and $P(t,T)$ without the time subscripts,

$$
\begin{aligned}
dV &= rVdt + \sigma_V V dW^Q && (44)\\
dP &= rPdt + \sigma_P P dW^Q. && (45)
\end{aligned}
$$

Note that since V and P are driven by the same Brownian motion, changes in V and P are perfectly correlated.

The objective is to determine the SDE for the function $F(V,P) = V/P$. First, we compute the requisite partial derivatives of F with respect to V and P.

$$
\begin{aligned}
\frac{\partial F(V,P)}{\partial V} &= \frac{1}{P} && (46)\\
\frac{\partial F(V,P)}{\partial P} &= \frac{-V}{P^2} && (47)\\
\frac{\partial^2 F(V,P)}{\partial V^2} &= 0 && (48)\\
\frac{\partial^2 F(V,P)}{\partial P^2} &= \frac{2V}{P^3} && (49)\\
\frac{\partial^2 F(V,P)}{\partial P \partial V} &= \frac{-1}{P^2}. && (50)
\end{aligned}
$$

Second, an application of Ito's lemma gives us

$$
\begin{aligned}
dF &= \left(\frac{1}{P}rV - \frac{V}{P^2}rP + \frac{V}{P^3}\sigma_P^2 P^2 - \frac{1}{P^2}\sigma_V V \sigma_P P\right) dt \; + \\
&\qquad \left(\frac{1}{P}\sigma_V V - \frac{V}{P^2}\sigma_P P\right) dW^Q && (51)\\
&= \left(\sigma_P^2 F - \sigma_V \sigma_P F\right) dt + \left(\sigma_V F - \sigma_P F\right) dW^Q && (52)\\
&= \sigma_P(\sigma_P - \sigma_V) F dt + (\sigma_V - \sigma_P) F dW^Q && (53)
\end{aligned}
$$

which is equation (9) in section 3. This completes the proof.

References

Fama, E.F. (1970), "Efficient Capital Markets: A Review of Theory and Empirical Tests," *Journal of Finance*, 25, 383–417.

Heath, D., R. Jarrow and A. Morton (1992), "Bond Pricing and the Term Structure of Interest Rates," *Econometrica*, 60, 77–105.

Jamshidian, F. (1989), "An Exact Bond Option Formula," *Journal of Finance*, 44, 205–209.

Jamshidian, F. (1991), "Bond and Option Evaluation in the Gaussian Interest Rate Model," *Research in Finance*, 9, 131–170.

Longstaff, F.A. (1995), "Hedging Interest Rate Risk with Options on Average Interest Rates," *Journal of Fixed Income*, March 1995, 37–45.

Lund, J. (1998), "Review of Continuous-Time Term-Structure Models — Part II: Arbitrage-Free Models," Lecture Notes, Department of Finance, Aarhus School of Business, April 1998.