Solution to “take home” problem 1

Question 1

The stochastic discount factor for real payoffs is

$$M_{t+1}^{real} = \delta \frac{U'(C_{t+1})}{U'(C_t)} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \tag{1}$$

Since the price deflator (index) at time t is D_t, the stochastic discount factor for nominal payoffs, M_{t+1}, is given by

$$M_{t+1} = M_{t+1}^{real} \frac{D_t}{D_{t+1}} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \frac{D_t}{D_{t+1}}, \tag{2}$$

from which the log stochastic discount factor, $m_{t+1} = \log M_{t+1}$, follows as

$$\begin{aligned} m_{t+1} &= \log \delta - \gamma \left(\log C_{t+1} - \log C_t \right) - \left(\log D_{t+1} - \log D_t \right) \\ &= \log \delta - \gamma u_{t+1} - \pi_t - v_{t+1}. \end{aligned} \tag{3}$$

Note that we have used equations (53) and (54) from assumptions A–2 and A–3 in the second line of (3).

Question 2

Since all innovations are normally distributed, we know that

$$\begin{aligned} P_{1t} &= E_t \left[M_{t+1} \right] \\ &= \exp \left[E_t(m_{t+1}) + \frac{1}{2} \text{Var}_t(m_{t+1}) \right] \\ &= \exp \left[-\pi_t + \log \delta + \frac{1}{2} \left(\gamma^2 \sigma_1^2 + \sigma_2^2 \right) \right], \end{aligned}$$

or

$$y_{1t} = -\log P_{1t} = \pi_t - \log \delta - \frac{1}{2} \left(\gamma^2 \sigma_1^2 + \sigma_2^2 \right). \tag{4}$$

Hence, since $y_{1t} \equiv \pi_t + \alpha$,

$$\alpha = -\log \delta - \frac{1}{2} \left(\gamma^2 \sigma_1^2 + \sigma_2^2 \right). \tag{5}$$

Question 3

This follows Campbell, Lo and MacKinlay, Section 11.1.1. First note that (assuming that the affine formula is correct)

$$\begin{aligned} m_{t+1} + p_{n-1,t+1} &= \log\delta - \gamma u_{t+1} - \pi_t - v_{t+1} \\ &\quad - A_{n-1} - B_{n-1} \underbrace{\left(\pi_t + \kappa(\theta - \pi_t) + w_{t+1}\right)}_{\pi_{t+1}} \end{aligned} \tag{6}$$

Next,

$$p_{nt} = E_t\left[m_{t+1} + p_{n-1,t+1}\right] + \frac{1}{2}\mathrm{Var}_t\left[m_{t+1} + p_{n-1,t+1}\right] \tag{7}$$

$$\begin{aligned} &= -\pi_t + \log\delta - A_{n-1} - B_{n-1}\left(\pi_t + \kappa(\theta - \pi_t)\right) \\ &\quad + \frac{1}{2}\left(\gamma^2\sigma_1^2 + \sigma_2^2 + B_{n-1}^2\sigma_3^2 + 2\gamma B_{n-1}\sigma_{13}\right) \end{aligned} \tag{8}$$

$$\begin{aligned} &= \log\delta - A_{n-1} - \kappa\theta B_{n-1} + \frac{1}{2}\left(\gamma^2\sigma_1^2 + \sigma_2^2\right) + B_{n-1}\gamma\sigma_{13} + \frac{1}{2}B_{n-1}^2\sigma_3^2 \\ &\quad - \left\{1 + (1-\kappa)B_{n-1}\right\}\pi_t \end{aligned}$$

$$= -A_n - B_n\pi_t \tag{9}$$

Since the terms match on both sides of the last equality (there is a coefficient for π_t, and one for the constant term), we have verified that the log of the bond price, p_{nt}, is an affine function of the state variable π_t, the expected inflation rate.

Question 4

Follows straightforwardly from the derivations involved in solving Question 3.

$$B_n = 1 + (1-\kappa)B_{n-1} \tag{10}$$

$$A_n = A_{n-1} - \log\delta + B_{n-1}\left(\kappa\theta - \gamma\sigma_{13}\right) - \frac{1}{2}\left(\gamma^2\sigma_1^2 + \sigma_2^2\right) - \frac{1}{2}B_{n-1}^2\sigma_3^2. \tag{11}$$

With initial conditions $A_0 = 0$ and $B_0 = 0$, as usual. The closed-form solution for B_n is (for $n \geq 1$)

$$B_n = \sum_{j=0}^{n-1}(1-\kappa)^j = \frac{1-(1-\kappa)^n}{\kappa}. \tag{12}$$

The first equality in (12) follows from recursive substitution of the difference equation (10), and the second by noting that the sum is the standard annuity growth formula with an "interest rate" of $-\kappa$.

Question 5

In general, we have the following relationship (see CLM 11.1.12) for the one-period excess holding period return:

$$E_t[r_{n,t+1} - y_{1t}] = -\mathrm{Cov}_t(r_{n,t+1}, m_{t+1}) - \frac{1}{2}\mathrm{Var}_t(r_{n,t+1}). \tag{13}$$

Since

$$r_{n,t+1} = p_{n-1,t+1} - p_{nt} = -A_{n-1} - B_{n-1}\pi_{t+1} - p_{nt}, \tag{14}$$

we get

$$\begin{aligned} E_t[r_{n,t+1} - y_{1t}] &= B_{n-1}\mathrm{Cov}_t(\pi_{t+1}, m_{t+1}) - \frac{1}{2}B_{n-1}^2\mathrm{Var}_t(\pi_{t+1}) \\ &= -B_{n-1}\gamma\sigma_{13} - \frac{1}{2}B_{n-1}^2\sigma_3^2. \end{aligned} \tag{15}$$

(here we have used $\mathrm{Cov}_t(\pi_{t+1}, m_{t+1}) = \mathrm{Cov}(w_{t+1}, -\gamma u_{t+1}) = -\gamma\sigma_{13}$).
Since the expected excess holding period return is independent of π_t, we may conclude that the EHT holds.

Question 6

First, the forward rate can be be written as

$$f_{nt} = (B_{n+1} - B_n)\pi_t + (A_{n+1} - A_n) = (1-\kappa)^n\pi_t + \mathrm{const}, \tag{16}$$

since

$$\begin{aligned} B_{n+1} - B_n &= 1 + (1-\kappa)B_n - B_n \\ &= 1 - \kappa B_n \\ &= 1 - \kappa\frac{1-(1-\kappa)^n}{\kappa} \\ &= (1-\kappa)^n. \end{aligned} \tag{17}$$

Second, note that according to the assumption in equation (54),

$$\log\frac{D_{t+n+1}}{D_{t+n}} = \pi_{t+n} + v_{t+n+1}. \tag{18}$$

Using these results, we obtain the following expression for the population value of the slope coefficient β_{1n} in the regression of the one-period inflation rate at time $t+n+1$

on the time t n-period forward rate:

$$\begin{aligned}\beta_{1n} &= \frac{\mathrm{Cov}\left(\log\frac{D_{t+n+1}}{D_{t+n}}, f_{nt}\right)}{\mathrm{Var}(f_{nt})}\\ &= \frac{\mathrm{Cov}\left(\pi_{t+n}+v_{t+n+1}, (1-\kappa)^n\pi_t\right)}{\mathrm{Var}\left((1-\kappa)^n\pi_t\right)}\\ &= \frac{(1-\kappa)^n(1-\kappa)^n\mathrm{Var}(\pi_t)}{(1-\kappa)^{2n}\mathrm{Var}(\pi_t)} \qquad (19)\\ &= 1,\end{aligned}$$

where we have used the following additional properties/assumptions

$$\mathrm{Cov}(\pi_{t+n},\pi_t) = (1-\kappa)^n\mathrm{Var}(\pi_t) \qquad (20)$$

$$\mathrm{Cov}(v_{t+n+1},\pi_t) = 0. \qquad (21)$$

A simpler way to deduce that the population value of the slope coefficient must be one is to note that $\pi_t = y_{1t}+\alpha$, and since v_{t+n+1} is independent of π_t, we are doing a regression of the short rate at time $t+n$ on the current (time t) n-period forward rate. Under the EHT, the forward rate is equivalent to the conditional expectation of the future short rate (apart from the constant term), so the regression coefficient must be one (in population value).

Question 7

The interpretation of having a slope of one in the regression (80) is that the n-period forward rate at time t is the optimal predictor of the inflation rate at time $t+n+1$.

Question 8

Nope.
Reason: in Question 5, the risk premium will be state-dependent (ruling out the EHT), and in Question 6, $f_{nt} = (1-\kappa)^n\pi_t + \text{const}$ no longer holds, and this property was needed to show that the regression slope had a population value of one.

Solution to “take home” problem 2

Question 1

The solution to the SDE (3) can be written as

$$r_{t+\Delta} = r_t + \int_t^{t+\Delta}\mu ds + \sigma\int_t^{t+\Delta} dW_s. \qquad (22)$$

Since the drift and diffusion coefficients are both constants, it is easily seen that the conditional distribution of $r_{t+\Delta}$ given r_t is the normal distribution with

$$\text{a)}\quad r_{t+\Delta} \sim \mathcal{N}\left(r_t + \mu\Delta,\, \sigma^2\Delta\right). \tag{23}$$

Under the risk-neutral measure (part b of the question), the SDE is

$$dr_t = (\mu - \lambda\sigma)dt + \sigma dW_t^Q. \tag{24}$$

Using the same arguments as above, we can show that the conditional distribution of $r_{t+\Delta}$ under the risk-neutral measure is

$$\text{b)}\quad r_{t+\Delta} \sim \mathcal{N}\left(r_t + (\mu - \lambda\sigma)\Delta,\, \sigma^2\Delta\right). \tag{25}$$

Question 2

The general form of the fundamental PDE for a one-factor model is

$$\frac{1}{2}\sigma^2(r)\frac{\partial^2 P}{\partial r^2} + \frac{\partial P}{\partial r}\left\{\mu(r) - \lambda(r)\sigma(r)\right\} + \frac{\partial P}{\partial t} - rP = 0. \tag{26}$$

In the particular case [the SDE in equation (66)], we get (since $\sigma(r)$, $\mu(r)$ and $\lambda(r)$ are all constants)

$$\frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} + \frac{\partial P}{\partial r}(\mu - \lambda\sigma) + \frac{\partial P}{\partial t} - rP = 0. \tag{27}$$

Note: In my opinion the above is sufficient to answer the question. I realize that the word “derive” in the question may be interpreted as requiring, say, the full derivation of why absence of arbitrage implies that the bond price must satisfy the PDE (26). However, this is not intended here. Needless to say, it is not an error to do so…

Question 3

Since the model is exponential-affine, we know that

$$P(t, t+\tau) = \exp[A(\tau) + B(\tau)r], \quad \tau = T - t. \tag{28}$$

The partial derivatives of (28) are

$$\frac{\partial P}{\partial r} = B(\tau)P \tag{29}$$

$$\frac{\partial^2 P}{\partial r^2} = B^2(\tau)P \tag{30}$$

$$\frac{\partial P}{\partial t} = -\frac{\partial P}{\partial \tau} = -\left[A(\tau) + B(\tau)r\right]P. \tag{31}$$

If we substitute these into the PDE, we get (after dividing all terms by P) the following:

$$\frac{1}{2}\sigma^2 B^2(\tau) + B(\tau)\left(\mu - \lambda\sigma\right) - A'(\tau) - B'(\tau)r - r = 0. \tag{32}$$

The next step is to collect all terms involving r and the ones involving a constant. This gives us two ODEs

$$\begin{aligned} B'(\tau) &= -1 && (33)\\ A'(\tau) &= \frac{1}{2}\sigma^2 B^2(\tau) + (\mu - \lambda\sigma)B(\tau), && (34) \end{aligned}$$

with the usual boundary conditions $B(0) = 0$ and $A(0) = 0$. The solutions are obtained by straightforward integration,

$$\begin{aligned} B(\tau) &= \int_0^\tau -1 ds = -\tau && (35)\\ A(\tau) &= \int_0^\tau \left\{\frac{1}{2}\sigma^2 s^2 - (\mu - \lambda\sigma)s\right\} ds \;=\; \frac{1}{6}\sigma^2\tau^3 - \frac{1}{2}(\mu - \lambda\sigma)\tau^2. && (36) \end{aligned}$$

We have now shown that the bond-price equation is

$$P(t, t+\tau) = \exp\left[\frac{1}{6}\sigma^2\tau^3 - \frac{1}{2}(\mu - \lambda\sigma)\tau^2 - \tau r_t\right]. \tag{37}$$

Question 4

The continuously compounded yield for maturity τ is defined by:

$$y(t, t+\tau) \;=\; -\log P(t, t+\tau)/\tau \;=\; r_t + \frac{1}{2}(\mu - \lambda\sigma)\tau - \frac{1}{6}\sigma^2\tau^2. \tag{38}$$

Note that

$$\begin{aligned} y(t+\Delta, \tau) - y(t, \tau) &= r_{t+\Delta} - r_t && (39)\\ y(t, \tau_2) - y(t, \tau_1) &= \frac{1}{2}(\mu - \lambda\sigma)(\tau_2 - \tau_1) - \frac{1}{6}\sigma^2(\tau_2^2 - \tau_1^2). && (40) \end{aligned}$$

Using the results obtained when answering Question 1 (see above), the expectations can be calculated as follows:

$$\begin{aligned} \text{a)}\ E_1(\psi) &= E\left[r_{t+\Delta} - r_t\right] \;=\; \mu\Delta && (41)\\ \text{b)}\ E_2(\psi) &= E\left[\left(r_{t+\Delta} - r_t\right)^2\right] \;=\; \left(E\left[r_{t+\Delta} - r_t\right]\right)^2 + \operatorname{Var}\left(r_{t+\Delta} - r_t\right) &&\\ &= \mu^2\Delta^2 + \sigma^2\Delta && (42)\\ \text{c)}\ E_3(\psi) &= E\left[(y(t, \tau_2) - y(t, \tau_1))\right. &&\\ &= \frac{1}{2}(\mu - \lambda\sigma)(\tau_2 - \tau_1) - \frac{1}{6}\sigma^2(\tau_2^2 - \tau_1^2). && (43) \end{aligned}$$

Actually, we take the expectation of a deterministic quantity in subquestion c) since the only randomness goes through r_t, which cancels out when using the difference between two yields at time t (this, by the way, is the reason for introducing measurement errors in the, admittedly, *ad hoc* way in Question 5).

Question 5

Since we need first-order differences to compute the sample moments, the first observation (t_1) is ignored, except to compute the requisite difference, that is $y(t_2,\tau_1)-y(t_1,\tau_1)$. This leaves $n=331$ effective observations (any other choice is equally good, including dropping the last observation, or using one additional observation for the third moment; doing so will generate small differences from the results below, however).

Define

$$m_k = \begin{bmatrix} y(t_{k+1},\tau_1)-y(t_k,\tau_1) \\ \left\{ y(t_{k+1},\tau_1)-y(t_k,\tau_1) \right\}^2 \\ y(t_{k+1},\tau_2)-y(t_{k+1},\tau_1) \end{bmatrix}, \tag{44}$$

and

$$\bar{m} = \frac{1}{n}\sum_{k=1}^{n} m_k \tag{45}$$

The sample mean is computed as:

$$\bar{m} = \begin{bmatrix} -8.6438\times 10^{-5} \\ 1.5188\times 10^{-6} \\ 8.5146\times 10^{-3} \end{bmatrix}. \tag{46}$$

The next step is to equalize the vector of sample moments (45) with the expected values calculated in Question 4.

$$\begin{aligned} \bar{m}_1 &= \mu\Delta & (47)\\ \bar{m}_2 &= \mu^2\Delta^2+\sigma^2\Delta & (48)\\ \bar{m}_3 &= \frac{1}{2}(\mu-\lambda\sigma)(\tau_2-\tau_1)-\frac{1}{6}\sigma^2(\tau_2^2-\tau_1^2). & (49) \end{aligned}$$

Solving these equations gives

$$\begin{aligned} \mu &= \bar{m}_1/\Delta & (50)\\ \sigma &= \sqrt{(\bar{m}_2-\mu^2\Delta^2)/\Delta} & (51)\\ \lambda &= \left(\mu - \frac{2\left\{\bar{m}_3+\frac{1}{6}\sigma^2(\tau_2^2-\tau_1^2)\right\}}{\tau_2-\tau_1}\right)\Big/\sigma & \\ &= \left(\mu-\frac{2\bar{m}_3}{\tau_2-\tau_1}-\frac{1}{3}\sigma^2(\tau_1+\tau_2)\right)\Big/\sigma. & (52) \end{aligned}$$

Inserting the sample means for $\bar{m}$ from equation (47), gives us the GMM estimates (the hat denotes “estimate”).

$$\begin{aligned} \hat{\mu} &= -0.004495\\ \hat{\sigma} &= 0.008865\\ \hat{\lambda} &= -1.6113 \end{aligned}$$

Question 6

In order to calculate asymptotic standard error, we need the Jacobian of the GMM moment conditions.

$$G = E\left[\frac{\partial f(y_k,\psi)}{\partial \psi'}\right] = -\frac{\partial E(\psi)}{\partial \psi'}, \tag{53}$$

where $E(\psi) = (\, E_1(\psi), E_2(\psi), E_3(\psi)\,)'$. Note: The notation $\partial E(\psi)/\partial \psi'$ means a matrix, whose (i,j)'th element is $\partial E_i(\psi)/\partial \psi_j$.

The non-zero elements of the 3×3 matrix G are

$$G = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{bmatrix}, \tag{54}$$

with

$$\begin{aligned}
g_{11} &= -\frac{\partial E_1(\psi)}{\partial \mu} &&= -\,\Delta && (55)\\
g_{21} &= -\frac{\partial E_2(\psi)}{\partial \mu} &&= -\,2\mu\Delta^2 && (56)\\
g_{22} &= -\frac{\partial E_2(\psi)}{\partial \sigma} &&= -\,2\sigma\Delta && (57)\\
g_{31} &= -\frac{\partial E_3(\psi)}{\partial \mu} &&= -\,\frac{1}{2}(\tau_2-\tau_1) && (58)\\
g_{32} &= -\frac{\partial E_3(\psi)}{\partial \sigma} &&= \frac{1}{2}\lambda(\tau_2-\tau_1) + \frac{1}{3}\sigma(\tau_2^2-\tau_1^2) && (59)\\
g_{33} &= -\frac{\partial E_3(\psi)}{\partial \lambda} &&= \frac{1}{2}\sigma(\tau_2-\tau_1). && (60)
\end{aligned}$$

The covariance matrix of the moment conditions, denoted S, is computed under the iid assumption (no autocorrelation). Under the maintained hypothesis that the model is correctly specified, this assumption is valid (whether it is supported by the data is another question; but if not, the whole model is cast into doubt, and we would only be improving on the standard errors for a misspecified model).

Hence, let

$$S = \frac{1}{n}\sum_{k=1}^{n}\left(m_k - E(\hat{\psi})\right)\left(m_k - E(\hat{\psi})\right)'. \tag{61}$$

Now, according to the standard GMM theory [see chapter 11 in Cochrane],

$$\mathrm{Cov}(\hat{\psi}) \;=\; \frac{1}{n}\left(G'S^{-1}G\right)^{-1} \;=\; \frac{1}{n}G^{-1}S\left(G^{-1}\right)', \tag{62}$$

since $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ and $(A')^{-1} = (A^{-1})'$, see any textbook on matrix algebra.

Note: in the above formula (62), we have assumed that the "optimal" weighting matrix S^{-1} is used. Since we have 3 moment conditions and 3 parameters, we know that the GMM estimates do not depend on the choice of weighting matrix. However, if we need to calculate standard errors with an arbitrary weighting matrix W, we cannot use the formula (62). Instead, we must use the following, more involved, formula:

$$\mathrm{Cov}(\hat{\psi}) = \left(G'WG\right)^{-1} G'WSWG \left(G'WG\right)^{-1}. \tag{63}$$

In the exactly identified case, where all matrices are square (and invertible), this simplifies to equation (62).

Here, the covariance matrix of the sample moments conditions can be computed as (only lower part is shown since the matrix is symmetric)

$$S = \left[\begin{array}{ccc} 1.5114 \times 10^{-6} & & \\ -1.711 \times 10^{-9} & 1.2147 \times 10^{-11} & \\ 8.2025 \times 10^{-7} & -1.1309 \times 10^{-9} & 2.8092 \times 10^{-5} \end{array}\right]. \tag{64}$$

The covariance matrix $\mathrm{Cov}(\hat{\psi})$ is computed as

$$\mathrm{Cov}(\hat{\psi}) = \left[\begin{array}{ccc} 1.2347 \times 10^{-5} & & \\ -6.681 \times 10^{-7} & 3.0146 \times 10^{-7} & \\ 1.2557 \times 10^{-3} & -1.9891 \times 10^{-5} & 1.3780 \times 10^{-1} \end{array}\right]. \tag{65}$$

Finally, by taking the square root of the diagonal elements, we get the desired results:

$$\begin{aligned} \mathrm{s.e.}(\hat{\mu}) &= 0.003514 \\ \mathrm{s.e.}(\hat{\sigma}) &= 0.0005491 \\ \mathrm{s.e.}(\hat{\lambda}) &= 0.3712 \end{aligned}$$